3 (Sem–3/CBCS) MAT HC 1

2023

MATHEMATICS

(Honours Core)

Paper : MAT–HC–3016

(Theory of Real Functions)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions : 1×10=10

   (a) Is 0 a cluster point of (0,1) ?

   (b) " If the limit of a function f at a point C of its domain does not exist, then f diverges at C." *(Write True or False)*

   (c) Define $\lim_{x \to c} f(x) = \infty$, where $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ and $C \in \mathbb{R}$ is a cluster point of A.

(d) Write sequential criterion for continuity.

(e) What do you mean by an unbounded function on a set ?

(f) Let $A, B \subseteq \mathbb{R}$ and let $f: A \to \mathbb{R}$ be continuous on A and let $g: B \to \mathbb{R}$ be continuous on B. Under what condition $g \circ f: A \to \mathbb{R}$ is continuous on A ?

(g) "If a function is continuous then it is uniformly continuous."

(Write True or False)

(h) If functions $f_1, f_2, \ldots, f_n$ are differentiable at c, write the expression for $(f_1.f_2.\ldots.f_n)'(c)$.

(i) The function $f(x) = x$ is defined on the interval $I = [0,1]$. Is 0 a relative maximum of f ?

(j) Define Taylor's polynomial for a function f at a point x_0, supposing f has an nth derivative at x_0.

2. Answer the following questions : 2×5=10

(a) Use $\varepsilon - \delta$ definition of limit to show that $\lim_{x \to 0} \frac{x^2}{|x|} = 0$.

(b) Show that the absolute value function $f(x) = |x|$ is continuous at every point $c \in \mathbb{R}$.

(c) Give an example of a function $f: [0,1] \to \mathbb{R}$ that is discontinuous at every point of $[0,1]$, but $|f|$ is continuous on $[0,1]$.

(d) "Continuity at a point is not a sufficient condition for the derivative to exist at that point." Justify your answer.

(e) Show that $\lim_{x \to 0+} \frac{\sin x}{\sqrt{x}} = 0$.

3. Answer **any four** parts : 5×4=20

(a) Prove that a number $c \in \mathbb{R}$ is a cluster point of a subset A of $\mathbb{R}$ if and only if there exists a sequence $\{a_n\}$ in A such that $\lim a_n = c$ and $a_n \neq c$ for all $n \in \mathbb{N}$.

(b) Show that (using $\varepsilon-\delta$ definition of limit)

$$\lim_{x\to 2}\frac{x^3-4}{x^2+1}=\frac{4}{5}$$

(c) Prove that if $I=[a,b]$ is a closed bounded interval and if $f:I\to\mathbb{R}$ is continuous on I then f is bounded on I.

(d) Show that if f and g are uniformly continuous on a subset A of $\mathbb{R}$ then $f+g$ is uniformly continuous on A.

(e) Suppose that f is continuous on a closed interval $I=[a,b]$ and that f has a derivative in the open interval (a,b). Then there exists *at least one* point c in (a,b) such that

$$f(b)-f(a)=f'(c)(b-a).$$

(f) Let $f:I\to\mathbb{R}$ be differentiable on the interval I. Then prove that f is increasing if and only if $f'(x)\geq 0$ for all $x\in I$.

4. Answer ***any four*** parts : 10×4=40

(a) Prove that a real valued function f is continuous at $c\in\mathbb{R}$ if and only if whenever every sequence $\{c_n\}$, converging to c, then corresponding sequence $\{f(c_n)\}$ converges to $f(c)$.

(b) (i) Show that every infinite bounded subset of $\mathbb{R}$ has *at least one* limit point. 5

(ii) Let $A\subseteq\mathbb{R}$, let $f:A\to\mathbb{R}$ and let $c\in\mathbb{R}$ be a cluster point of A. If $a\leq f(x)\leq b$ for all $x\in A$, $x\neq c$ and if $\lim_{x\to c} f(x)$ exist then prove that

$$a\leq \lim_{x\to c} f\leq b.$$ 5

(c) (i) Let $I=[a,b]$ be a closed bounded interval. Let $f:I\to\mathbb{R}$ be such that f is continuous. Prove that f is uniformly continuous on $[a,b]$. 5

(ii) Show that the function $f(x)=\frac{1}{x^2}$ is uniformly continuous on $A=[1,\infty)$. 5

(d) Let $I=[a,b]$ be a closed bounded interval and let $f:I\to\mathbb{R}$ be continuous on I. Then f has an absolute maximum and an absolute minimum on I.

(e) (i) Let I be a closed bounded interval and let $f:I\to\mathbb{R}$ be continuous on I. Then the set $f(I)=\{f(x):x\in I\}$ is a closed bounded interval. 6

(ii) Let $A,B\subseteq\mathbb{R}$ and let $f:A\to\mathbb{R}$ and $g:B\to\mathbb{R}$ be functions such that $f(A)\subseteq B$. If f is continuous at a point $c\in A$ and g is continuous at $b=f(c)\in B$, then show that the composition $g\circ f:A\to\mathbb{R}$ is continuous at c. 4

(f) (i) Let $I=[a,b]$ and let $f:I\to\mathbb{R}$ be continuous on I. If $f(a)<0<f(b)$ or if $f(a)>0>f(b)$, then prove that there exists a number $c\in(a,b)$ such that $f(c)=0$. 6

(ii) Use the definition to find the derivative of the function $f(x)=\frac{1}{\sqrt{x}}$ for $x>0$. 4

(g) (i) State and prove Taylor's theorem. 2+5=7

(ii) Using the Mean Value theorem prove that $|\sin x-\sin y|\le|x-y|$ for all x, y in $\mathbb{R}$. 3

(h) (i) Show that

$$1-\frac{1}{2}x^2\le\cos x$$

for all $x\in\mathbb{R}$ 5

(ii) Evaluate $\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)$ 5